Accuracy and Integrity of Nonlinear Systems

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BIOGRAPHY

Bastiaan Ober's areas of experience include the influence of multipath on GPS positioning, carrier phase differential GPS, ambiguity resolution and integrity monitoring, especially for aviation applications. He is currently working as a Ph.D. student doing research on integrity design and analysis of integrated navigation systems for safety critical applications.

ABSTRACT

Although measurements and position are often related in a non-linear fashion, system models and computations usually exploit a linearized model. The paper provides insight in the amount of error that is introduced by this approximation. It shows why and how extra errors are introduced, estimates the sizes of these errors and quantifies the influence on system accuracy and integrity.

1. INTRODUCTION

Although in most systems the relation between measurements and position is nonlinear, the assumption that the relation between the position and the measurements is sufficiently well described by a linear regression model is widespread. Throughout navigation literature, the linear model plays an important role, as an advertisement of the course "Aviation system performance analysis using the linear model" [Kelly] shows:

> *All standard documents employ the linear model as their common conceptual basis. Specs and requirements are defined in this context. One must understand the linear model in order to comprehend GPS monitoring and many other operations. Even more important, each user will be required by the FAA to demonstrate its compliance with FAA standards in terms of a linear model.*

Thus, even though the linear model is only an approximation, it serves as a reference point within the navigation community. A major reason for the popularity of the linear model is the availability of a well-developed theory, which provides insights that are valuable even in case the model is not fully accurate. Of course, it remains important to study the validity of the assumption that the results obtained with the linear model describe the non-linear model it approximates 'sufficiently well' in some sense.

The paper investigates to what extend a nonlinear system be described by this linearized model. It can be seen as an attempt to quantify the gap between the nonlinear model and its linear approximation in terms of both accuracy and integrity.

The approach taken is straightforward. The position is determined by non-linear least-squares. Instead of using a linear approximation to describe the properties of the position estimation that is obtained, a second order quadratic approximation is exploited and used as a 'truth-reference' to assess the correctness of the linear model. It is shown that the non-linear case can suffer from an increased sensitivity to noise and biases, and can in fact be written in a linear form with a non-linearly increased amount of noise.

2. THE LINEARIZED SYSTEM MODEL

We will assume in this paper that the relation between the n measurements $\underline{z}$ and the m unknowns $\underline{u}$, that can for example represent the position, are given by:

$$\underline{z} = h(\underline{u}) + \underline{n} \tag{2.1}$$

in which $\underline{n}$ represents the measurement noise and errors. We will use the usual assumption that the noise is normally distributed with mean $\underline{\mu}_n$ and covariance Σ_n. In the absence of system failures $\underline{\mu}_n = \underline{0}$. When distributions are centered on zero, we will call them unbiased. When they are not, they are biased and the mean is sometimes referred to as the *bias* in the distribution.

For a given set of measurements, a least squares estimated solution $\hat{\underline{u}}$ can be found by minimizing the norm of the residual $\underline{r}$:

$$\underline{r} = \underline{z} - h(\hat{\underline{u}}) \tag{2.2}$$

Such a minimum norm is readily found by the Gauss-

Newton method, also known as the linearization method, although other methods might be applied as well [Ratkowsy83]. Given an initial guess $\underline{\hat{u}}_0$ of the unknowns, a linear approximation of h is obtained as the first order Taylor approximation:

$$h(\underline{u}) = h(\underline{\hat{u}}_0) - H_{\underline{\hat{u}}_0} \Delta \underline{u}_0 \tag{2.3}$$

in which the position estimation error is written as:

$$\Delta \underline{u}_0 = \underline{\hat{u}}_0 - \underline{u} \tag{2.4}$$

The $n \times m$ matrix $H_{\underline{\hat{u}}_0}$ contains the first derivatives of the elements of $\underline{z}$ to the elements of $\underline{u}$ and is defined as

$$H_{\underline{x}} = \begin{bmatrix} \frac{\partial h[1]}{\partial x[1]} & \cdots & \frac{\partial h[1]}{\partial x[m]} \\ \vdots & \ddots & \vdots \\ \frac{\partial h[n]}{\partial x[1]} & \cdots & \frac{\partial h[n]}{\partial x[m]} \end{bmatrix} \tag{2.5}$$

where $[i]$ denotes the i^{th} element of each vector. Substituting this in (2.2) gives a linear expression for the residual:

$$\underline{r}_0 = \underline{z} - h(\underline{\hat{u}}_0) = H_{\underline{\hat{u}}_0} (\underline{u} - \underline{\hat{u}}_0) + \underline{n} \tag{2.6}$$

It is well known [Rao95] that the residual $\underline{r_0}$ is minimised for $\underline{u} = \underline{\hat{u}}_1$ with

$$\underline{\hat{u}}_1 - \underline{\hat{u}}_0 = H^+_{\underline{\hat{u}}_0} (\underline{z} - h(\underline{\hat{u}}_0)) \tag{2.7}$$

in which the '+' denotes the pseudo-inverse:

$$H^+_{\underline{\hat{u}}_0} = (H^T_{\underline{\hat{u}}_0} \Sigma_n^{-1} H_{\underline{\hat{u}}_0})^{-1} H^T_{\underline{\hat{u}}_0} \Sigma_n^{-1} \tag{2.8}$$

Substituting $\underline{\hat{u}}_1$ in (2.2) again, an improved linear approximation of h is obtained, a better estimator can be derived, again leading to a better linearization…. The iteration ends when the estimated value doesn't change anymore and the so-called likelihood equations are obeyed:

$$H^+_{\underline{\hat{u}}_0} (\underline{z} - h(\underline{\hat{u}})) = \underline{0} \tag{2.9}$$

The estimation deviation and the residual now obey the following linear relations:

$$\Delta \underline{u} = \underline{\hat{u}} - \underline{u} = H^+_{\underline{\hat{u}}} \underline{n} \tag{2.10}$$

and

$$\underline{r} = (I - H_{\underline{\hat{u}}} H^+_{\underline{\hat{u}}}) \underline{n} \tag{2.11}$$

2.1 Accuracy and integrity within the linear model

Performance analysis is usually based on the linear relations (2.10) and (2.11): to assess the accuracy of the position estimation, the covariance propagation rules are used to derive the position estimation covariance $\Sigma_{\hat{u}}$ from the noise covariance Σ_n as

$$\Sigma_{\hat{u}} = H^+_{\underline{\hat{u}}} \Sigma_n H^{+T}_{\underline{\hat{u}}} \tag{2.12}$$

Similarly, the covariance of the residuals equals

$$\Sigma_r = (I - H_{\underline{\hat{u}}} H^+_{\underline{\hat{u}}}) \Sigma_n (I - H_{\underline{\hat{u}}} H^+_{\underline{\hat{u}}})^T \tag{2.13}$$

For integrity, performance measures are often based on the propagation of measurement biases into both the position estimation and the residual, which is used for bias detection. When the measurement bias is $\underline{\mu}_z$, the position and residual biases simply become

$$\underline{\mu}_{\Delta u} = H^+_{\underline{\hat{u}}} \underline{\mu}_n \tag{2.14}$$

and

$$\underline{\mu}_r = (I - H_{\underline{\hat{u}}} H^+_{\underline{\hat{u}}}) \underline{\mu}_n \tag{2.15}$$

A measure of integrity is the maximum ratio between this 'mean position error' and 'mean bias detection statistic':

$$SLOPE(\underline{\mu}_z) = \frac{\left\| \underline{\mu}_{\hat{u}} \right\|}{\left\| \underline{\mu}_r \right\|} \tag{2.16}$$

Often, the bias vector $\underline{\mu}_z$ is of the form

$$\vec{\mu}_z^{(i)} = \begin{bmatrix} \underset{1}{0} \cdots \underset{i-1}{0} & \underset{i}{\mu} & \underset{i+1}{0} \cdots \underset{n}{0} \end{bmatrix}^T \tag{2.17}$$

corresponding to a single failure in the i^{th} measurement. Integrity can then be represented by the maximum value of (2.16) over all possible single failure situations:

$$SLOPE_{MAX} = \max_i = \frac{\left\| \underline{\mu}_{\hat{u}} (\underline{\mu}_z^{(i)}) \right\|}{\left\| \underline{\mu}_r (\underline{\mu}_z^{(i)}) \right\|} \tag{2.18}$$

Note that both the value of (2.16) as (2.18) do not depend on the actual size of the bias, and can thus be computed without knowledge of this bias.

Sometimes, other integrity measures that are closely related to these slopes are used. In this paper, we will use the so-called protection level (*PL*) that equals the maximum slope times the minimal detectable bias (*MDB*), the smallest bias that can be detected with sufficient probability (for details see [Leva96], where the *MDB* is called *pbias*):

$$PL = SLOPE_{MAX} \cdot MDB \tag{2.19}$$

3. SECOND ORDER SYSTEM MODEL

The approach we will take to investigate the quality of the linear system approximation is simple. We will exploit second instead of first order Taylor approximations of $h(\underline{u})$. Using these second order approximations, we derive expressions for the bias and covariance of the estimated values and their residuals.

In fact, two second order Taylor approximations will be required, one around an estimated solution $\hat{\underline{u}}$ and one around the true solution $\underline{u}$:

$$\begin{aligned} h(\underline{u}) &\approx h(\hat{\underline{u}}) - H_{\hat{\underline{u}}}\Delta\underline{u} + \tfrac{1}{2}\Delta\underline{u}^T Q\Delta\underline{u} \\ h(\hat{\underline{u}}) &\approx h(\underline{u}) + H_{\underline{u}}\Delta\underline{u} + \tfrac{1}{2}\Delta\underline{u}^T Q\Delta\underline{u} \end{aligned} \tag{3.1}$$

When the quadratic approximation of *h* is valid, its second derivative *Q* is a constant 3-dimensional array that is independent of the value of $\underline{u}$. It is convenient to view *Q* as *n* stacked matrices Q_i, each containing the second derivatives of *h[i]* with respect to the elements of $\underline{u}$:

$$Q_i = \begin{bmatrix} \dfrac{\partial^2 \underline{h}[i]}{\partial^2 \underline{u}[1]} & \cdots & \dfrac{\partial^2 \underline{h}[i]}{\partial \underline{u}[1]\partial \underline{u}[m]} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \underline{h}[i]}{\partial \underline{u}[m]\partial \underline{u}[1]} & \cdots & \dfrac{\partial^2 \underline{h}[n]}{\partial^2 \underline{u}[m]} \end{bmatrix} \tag{3.2}$$

In the following, a few simple rules for expressions involving the matrix stack *Q* will be used. For matrices *A* and *B* and vectors $\underline{a}$ and $\underline{b}$ of appropriate sizes:

$$\begin{aligned} &AQB \text{ is an array of } n \text{ stacked matrices } AQ_iB \\ &\underline{a}^T Q\underline{b} \text{ is a vector } \left[\underline{a}^T Q_1\underline{b} \quad \cdots \quad \underline{a}^T Q_n\underline{b}\right]^T \end{aligned} \tag{3.3}$$

Furthermore, the trace of a three dimensional array is defined as

$$tr(Q) = \left[tr(Q_1) \quad \cdots \quad tr(Q_n)\right]^T \tag{3.4}$$

We now have all the tools available to obtain expressions for the position error $\Delta\underline{u}$. Starting point is the residual, that using the second order expansion of $h(\hat{\underline{u}})$ can be written as

$$\underline{r} = \underline{n} - H_{\underline{u}}\Delta\underline{u} - \tfrac{1}{2}\Delta\underline{u}^T Q\Delta \tag{3.5}$$

Because the true position is unknown, it is preferable to use the following relation, obtained by adding the two equations of (3.1):

$$H_{\underline{u}}\Delta\underline{u} \approx H_{\hat{\underline{u}}}\Delta\underline{u} - \Delta\underline{u}^T Q\Delta\underline{u} \tag{3.6}$$

and write

$$\underline{r} = \underline{n} - H_{\hat{\underline{u}}}\Delta\underline{u} + \tfrac{1}{2}\Delta\underline{u}^T Q\Delta \tag{3.7}$$

Substitution in the likelihood equations gives:

$$H_{\hat{u}}^T \Sigma_n^{-1}(\underline{n} - H_{\hat{\underline{u}}}\Delta\underline{u} + \tfrac{1}{2}\Delta\underline{u}^T Q\Delta) = \underline{0} \tag{3.8}$$

and after multiplication with $(H_{\hat{u}}^T \Sigma_n^{-1} H_{\hat{u}})^{-1}$

$$\Delta\underline{u} = H_{\hat{u}}^+ \underline{n} + \tfrac{1}{2} H_{\hat{u}}^+ \Delta\underline{u}^T Q\Delta\underline{u} \tag{3.9}$$

In accordance with [Seber89], we will eliminate the quadratic term in the estimation error by substituting the linear approximation $\Delta\underline{u} = H_{\hat{\underline{u}}}^+ \underline{n}$. We finally arrive at the following expression for the estimation error:

$$\Delta\underline{u} = H_{\hat{\underline{u}}}^+ (\underline{n} + \underline{n}^T A\underline{n}) \tag{3.10}$$

with

$$A = \tfrac{1}{2} H_{\hat{\underline{u}}}^{+T} Q H_{\hat{\underline{u}}}^+ \tag{3.11}$$

We see that the nonlinearity of the system has the same effect as the introduction of some extra noise. When we define a modified noise term as:

$$\underline{n}' = \underline{n} + \underline{n}^T A\underline{n} \tag{3.12}$$

the expressions for the estimation error and the residual can conveniently be written in their usual forms (cf. (2.10) and (2.11)) as:

$$\Delta\underline{u} = H_{\hat{\underline{u}}}^+ \underline{n}' \tag{3.13}$$

and

$$\underline{r} = (I - H_{\hat{\underline{u}}} H_{\hat{\underline{u}}}^+)\underline{n}' \tag{3.14}$$

The modified noise term acts as a kind of 'experienced noise' that incorporates both measurement noise and non-linearity effects.

3.1 Accuracy and integrity in the second order model

The mean and covariance of the position estimation and residual are easy to compute from their linear relations with $\underline{n}'$, just like has been done in paragraph 2.1 for $\underline{n}$. The same is true for the integrity related parameters (2.18) and (2.19).

All that remains to be done is the determination of the mean and covariance of the modified noise vector. We will only give results here, without the lengthy proof (see for example [Mathai92]). The mean modified noise is

$$\underline{\mu}_{n'} = \underline{\mu}_n + \underline{\mu}_n^T A \underline{\mu}_n + tr(A\Sigma_n) \tag{3.15}$$

while the elements of its covariance equal:

$$\begin{aligned}\Sigma_{n'}[i][j] = \Sigma_n[i][j] + 2(\underline{e}_i^T \Sigma_n A_j + \underline{e}_j^T \Sigma_n A_i)\underline{\mu}_n \\ + 2tr(\Sigma_n A_i \Sigma_n A_j) + 4\underline{\mu}_n^T A_i^T \Sigma_n A_j \underline{\mu}_n\end{aligned} \tag{3.16}$$

It is important to note that due to the non-linearity of the system, normally distributed measurements will not lead to normally distributed estimation errors and residuals. Therefore, mean and covariance do not fully describe their distributions. However, taking higher order moments into account would make the analysis much more complicated, and is beyond the scope of this paper. When substantial influence of non-linearity is found, this might justify future studies in this direction.

3.2 Comparing the models

In this section, we will summarize the conclusions that can be drawn from the previous analysis. Looking at the no failure case, in which the measurements are unbiased and $\underline{\mu}_n = \underline{0}$, we can see that:

$$\underline{\mu}_{\Delta u}^{(2)} = H_{\underline{\hat{u}}}^{+} tr(A\Sigma_n) \tag{3.17}$$

in which the superscript $^{(2)}$ denotes 'second order'. The position estimation is therefore biased even when the measurements are not. An explicit expression for the covariance of the position estimation error can easily be derived from (3.16) but we will avoid confusing the reader by writing it in explicit form here.

From (3.16) we can see that even when the measurements are uncorrelated, the non-linearity acts as if the measurements were in translating the measurement to the position noise Furthermore, the elements of the measurement covariance matrix all grow:

$$\Sigma_n^{(2)}[i][j] - \Sigma_n^{(1)}[i][j] = 2tr(\Sigma_n A_i \Sigma_n A_j) \tag{3.18}$$

and therefore, so will the covariance of the position estimation error.

The situation becomes a bit more complicated when there are measurement failures. The error in the estimation bias becomes larger:

$$\underline{\mu}_{\Delta u}^{(2)} - \underline{\mu}_{\Delta u}^{(1)} = H_{\underline{\hat{u}}}^{+}\left(tr(A\Sigma_n) + \underline{\mu}_n^T A \underline{\mu}_n\right) \tag{3.19}$$

and the changes in the 'experienced measurement covariance' get more pronounced.

For integrity, it is also important to assess the effect of the bias on the residual that is used for error detection. The error made in the linear approximation can be expressed as:

$$\underline{\mu}_r^{(2)} - \underline{\mu}_r^{(1)} = (I - H_{\underline{\hat{u}}} H_{\underline{\hat{u}}}^{+})\left(tr(A\Sigma_n) + \underline{\mu}_n^T A \underline{\mu}_n\right) \tag{3.20}$$

For the integrity measure of (2.19) we can conclude, that possibly both nominator and denominator become larger due to the non-linearity. Also, the extra noise that is experienced, will require lowering of the threshold to be set for error detection. The overall result of these different effects is hard to predict; but it seems that the non-linearity can possibly both worsen and improve integrity.

4. AN EXAMPLE

The expressions for second order system approximation have been incorporated in NavSim, the navigation system simulator that is being developed at Delft University. Although the algorithms are not yet fully verified, first tests indicate they seem to work correctly; however, full verification will only be performed shortly after the ION-GPS99 conference.

Because of the insight they might provide, it might still be useful to present some preliminary results. In a first trial, the *HDOP*, horizontal position bias and horizontal protection level (*HPL*) have been computed for the GPS position integrity performance requirements from [DO-208], from which all GPS system parameters are taken as well.

For three random points, 12 hours of data have been collected with 2 minutes between the samples from stand-alone GPS. The results are summarized in Table 1.

Accuracy has been measured in terms of both the position bias and the *HDOP* within both the linear and second order model. It seems hardly affected: the maximum position bias is in the sub-millimeter range, and the increase in *HDOP* is hardly visible.

Integrity has been measured in terms of the *HPL*. First, the first order model has been used to determine the worst case satellite (that reaches maximum *SLOPE* in (2.18)) and the size of the minimal detectable bias *MDB* on this satellite. The *MDB* is than used for substitution in (2.18) and (2.19) to determine the *SLOPE* and the *HPL* for the second order model. So, the ratio between the second and first order *HPL* is obtained at a specific value of the measurement bias only. This method does not give exact results: it might be that in the second order model another satellite becomes the worst case satellite. Furthermore, the *MDB* within the second order model might differ from the value used here (but can only be determined iteratively). The results indicate that on average the protection level decreases, although it improves in some cases. For the third simulation, the scatterplot of the second order *HPL* against the first order *HPL* is given in Figure 1. Most of the time, the second order *HPL* is significantly smaller than the first order *HPL*.

5. CONCLUDING REMARKS

Preliminary simulations seem to indicate that for stand alone GPS accuracy is hardly influenced by the presence of non-linearity. It seems that in the absence of measurement failures, the system is almost linear. Integrity however, seems seriously influenced. The results definitely justify more extensive research to use the theoretical framework presented in this paper to give more definite statements about the integrity of existing and future systems.

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Table 1. Second order against linear approximation

$DOP^{(2)}/DOP^{(1)}$	Bias in m	$PL^{(2)}/PL^{(1)}$		
Max	Max	Min	Mean	Max
1.0000	0.00015	0.02	0.76	1.58
1.0000	0.000071	0.10	0.81	2.88
	0.0000116	0.15	0.70	1.69

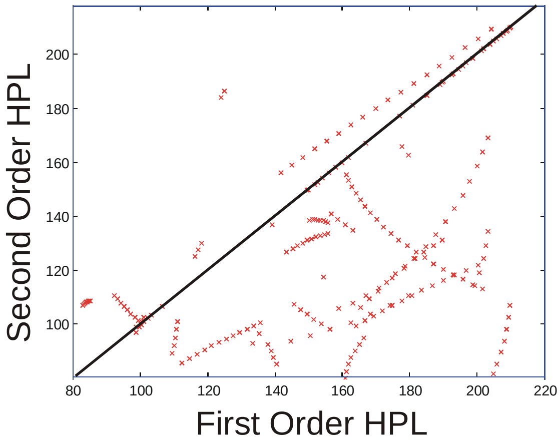

Figure 1. *The second order horizontal protection level against the first order case. The black line indicated equality of these levels; points under the line imply a smaller protection level when second order approximation is used as compared to the linear approximation.*